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NOTE

DENSITY RESULTS FOR UNIFORM FAMILIES

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A family $\mathcal{F}$ of subsets is called *k-dense* if there exists a k-element set A such that all 2^k of its subsets can be obtained in the form $A\cap F$ for some $F\in\mathcal{F}$. Best possible bounds are obtained for the maximum number of sets in not k-dense k-partite families. This is a consequence of a new rank formula for inclusion matrices.

1. Introduction

Let X be a finite set, $X=\{1,2,\dots,n\}$. A family $\mathcal{F}\subset 2^X$ is called *ℓ-dense*, ℓ a positive integer, if there exists an ℓ-element subset D such that for *all* subsets B of D there exists some $F=F(B)\in\mathcal{F}$ with $D\cap F=B$. That is,

$$\mathcal{F}_D := \{F\cap D : F\in\mathcal{F}\} = 2^D.$$

It is a classical result of Sauer [7], Shelah–Perles [8] and Vapnik-Červonenkis [9] that

$$\text{If } |\mathcal{F}| > \sum_{i<\ell}\binom{n}{i}, \text{ then } \mathcal{F} \text{ is } \ell\text{-dense.} \tag{1}$$

For simple proofs see [1], [3] and [5]. Let us mention that the maximal ℓ for which $\mathcal{F}$ is ℓ-dense is usually called the Vapnik-Červonenkis dimension of $\mathcal{F}$.

A family $\mathcal{F}$ is called *k-uniform* if $|F|=k$ holds for all $F\in\mathcal{F}$. Also, $\mathcal{F}$ is called *uniform* if it is k-uniform for some k.

In [5] it is proved that (1) can be improved for uniform families.

$$\text{If } |\mathcal{F}| > \binom{n}{\ell-1}, \text{ and } \mathcal{F} \text{ is uniform then } \mathcal{F} \text{ is } \ell\text{-dense.} \tag{2}$$

Considering the families $\binom{X}{\ell-1}$ or $\binom{X}{n-\ell+1}$ shows that (2) is best possible.

Let us recall the following conjecture.

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Conjecture 1.. [5] *Suppose that $\mathcal{F} \subset \binom{X}{k}$ is not k-dense, $n > 2k$ then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1} \tag{3}$$

holds.

Note that if true, then (3) sharpens the Erdős-Ko-Rado Theorem ([2]), which asserts that (3) holds under the much stronger assumption: $\mathcal{F}$ contains no pair of disjoint sets, i.e., $\Phi \notin \mathcal{F}_F$ holds for every $F \in \mathcal{F}$, and obviously, $H \notin \mathcal{F}_H$ for $H \in \left(\binom{X}{k} - \mathcal{F}\right)$.

Unfortunately, we could not prove (3), but we shall derive best possible bounds for an important special class of families.

Suppose that $X = X_1 \cup X_2 \cup \ldots \cup X_k$ is a partition and $\mathcal{F} \subset \binom{X}{k}$ satisfies $|F \cap X_i| = 1$ for all $F \in \mathcal{F}$ and $1 \leq i \leq k$. Then $\mathcal{F}$ is called *k-partite* and $X_1 \cup X_2 \cup \ldots \cup X_k$ is the *corresponding partition* (there can be several corresponding partitions for the same $\mathcal{F}$). Let $\mathcal{K}(X_1, \ldots, X_k)$ be the *complete k-partite family* that is,

$$\mathcal{K}(X_1, \ldots, X_k) = \{F \in \binom{X}{k} : |F \cap X_i| = 1, \quad 1 \leq i \leq k\}.$$

Note that $|\mathcal{K}(X_1 \ldots, X_k)| = |X_1| \cdot \ldots \cdot |X_k|$.

Example 1. Let $X = X_1 \cup \ldots \cup X_k$ be a partition and let $A = \{x_1, \ldots, x_k\}$ be a set satisfying $x_i \in X_i$ for $1 \leq i \leq k$.

Define $\mathcal{B} = \mathcal{B}(X_1, \ldots, X_k)$ by

$$\mathcal{B} = \{B \in \mathcal{K}(X_1, \ldots, X_k), \quad B \cap A \neq \Phi\}.$$

Now, $\mathcal{B}$ is not k-dense. Indeed, let $F \in \binom{X}{k}$ be arbitrary. If $|F \cap X_i| \geq 2$ for some i, then $F \cap B \neq F \cap X_i$ is obvious for all $B \in \mathcal{B}$. Otherwise the set $F - A$ is missing from $\mathcal{B}_F$, because k-partiteness and $B \cap F = F - A$ would imply $B \cap A = \Phi$, i.e., $B \notin \mathcal{B}$.

Theorem 1. *Suppose that $\mathcal{F} \subset \mathcal{K}(X_1, \ldots, X_k)$ is a k-partite, not k-dense family. Then*

$$|\mathcal{F}| \leq |\mathcal{B}(X_1, \ldots, X_k)| \tag{4}$$

holds.

We shall prove (4) using a rank formula. For a matrix M let $\text{rank}_p M$ denote its rank over the field of p elements.

For a family $\mathcal{F} \subset \binom{X}{k}$ and an integer $0 \leq \ell \leq k$ one defines the *inclusion matrix of order ℓ, $M(\ell, \mathcal{F})$* as an $\binom{n}{\ell}$ by $|\mathcal{F}|$ matrix with columns indexed by the sets $F \in \mathcal{F}$, and the rows by $G \in \binom{X}{\ell}$, and the general entry $m(G, F)$ being

$$m(G, F) = \begin{cases} 1, & \text{if } G \subset F \\ 0, & \text{if } G \not\subset F. \end{cases}$$

Theorem 2.

$$\mathrm{rank}_p M(k-1, \mathcal{K}(X_1,\dots,X_k)) = |\mathcal{B}(X_1,\dots,X_k)| \tag{5}$$

holds for every p.

Call a family $\mathcal{F}$ *ℓ-degenerate* if there exists a numbering of its members $\mathcal{F} = \{F_1,\dots,F_m\}$ and a collection $\mathcal{G} = \{G_1,\dots,G_m\}$ of ℓ-element sets such that

$$G_i \subset F_i, \quad 1 \le i \le m \text{ and}$$
$$G_i \not\subset F_j, \quad 1 \le i < j \le m \text{ hold.}$$

Considering the rows of the inclusion matrix $M(\ell,\mathcal{F})$ corresponding to $\mathcal{G}$, we obtain a lower triangular matrix, with an all-one diagonal. This implies rank $M(\ell,\mathcal{F}) = |\mathcal{F}|$ over any field. Therefore, via the following proposition, (5) implies (4).

Proposition 1. *Suppose that $\mathcal{F} \subset \binom{X}{k}$ is a k-partite, not a k-dense family. Then $\mathcal{F}$ is $(k-1)$-degenerate.*

The paper is organized as follows. In Section 2 the proof of Theorem 2 and in Section 3 the proof of Proposition 1 is given. In the last section related and open problems are mentioned.

2. The proof of the rank formula (5).

Consider an ordering $B_1, B_2, \dots, B_m$ of the members of $\mathcal{B} = \mathcal{B}(X_1,\dots,X_k)$, i.e., $m = |\mathcal{B}|$, satisfying $|B_i \cap A| \le |B_j \cap A|$ for $1 \le i < j \le m$. Consequently, $B_m = A$.

Define $A_i = B_i - A$ and note that

$$A_i \not\subset B_j \text{ holds for } 1 \le i < j \le m. \tag{6}$$

Let D_i be an arbitrary $(k-1)$-element set satisfying $A_i \subset D_i \subset B_i$, $1 \le i \le m$.

From (6) it follows that the submatrix of $M = M(k-1, \mathcal{K}(X_1,\dots,X_k))$ spanned by the rows indexed by D_i, and the columns indexed by B_j, is an lower triangular matrix in which every entry of the main diagonal is 1. Consequently, the rank of M is at least $|\mathcal{B}|$ over *any* field. This proves the lower bound part of (5). To prove the upper bound part we are going to show that if one adds any column vector to the column vectors indexed by $B \in \mathcal{B}$, the vectors cease to be linearly independent over any field.

Indeed suppose that the new column is indexed by C. Since $C \notin \mathcal{B}$, $C \cap A = \Phi$. Also, C is the *only* k-subset of the $2k$-element set $A \cup C$ which is disjoint to A. Therefore the family $\mathcal{G} = \{G \in \mathcal{K}(X_1,\dots,X_k) : G \subset A \cup C\}$ satisfies $\mathcal{G} \subset (\mathcal{B} \cup \{C\})$.

Note that $|\mathcal{G}| = 2^k$. We claim that by summing the column vectors indexed by $G \in \mathcal{G}$ and with coefficients $(-1)^{|G \cap A|}$ one obtains the all-zero vector. Indeed, let H be an arbitrary $(k-1)$-element set. If H is contained in some $G \in \mathcal{G}$, then it is contained in exactly two, say G_1, G_2. Moreover, from the extra elements $G_1 - H$, $G_2 - H$ one is in A, the other is in C. Consequently, $(-1)^{|G_1 \cap A|} = -(-1)^{|G_2 \cap A|}$ holds, proving our claim. ∎

3. The proof of Proposition 1.

Suppose that $\mathcal{F} \subset \mathcal{K}(X_1, \dots, X_k)$ is not $(k-1)$-degenerate. Set $\mathcal{F}_0 = \mathcal{F}$. Suppose that $\mathcal{F}_i$ was defined for $i = 0, 1, \dots, s$. If there exists some $F \in \mathcal{F}_s$, and $G \in \binom{F}{k-1}$ such that $G \subset F' \in \mathcal{F}_s$ implies $F' = F$, i.e., F is the only member of $\mathcal{F}_s$ containing G. Then set $F_{s+1} = F$, $G_{s+1} = G$ and $\mathcal{F}_{s+1} = \mathcal{F}_s - \{F\}$.

Since $\mathcal{F}$ is not $(k-1)$-degenerate, this procedure will stop. That is, we obtain a non-empty family $\mathcal{F}_t \subset \mathcal{F}$ such that for every $F \in \mathcal{F}_t$ and every $G \in \binom{F}{k-1}$ there is some $F \neq F' \in \mathcal{F}_t$ with $G \subset F'$.

We are going to conclude the proof by showing that $\mathcal{F}_{t|F} = 2^F$ holds, that is, $\mathcal{F}_t$ is k-dense in a strong sense.

Suppose for contradiction, that for some $H \subset F$ there is no $F(H) \in \mathcal{F}_t$ with $F \cap F(H) = H$. Choose H in a way that $|H|$ is maximal with respect to this property. Since $F \cap F = F$, $|H| \leq k-1$ holds.

Let K be an arbitrary set satisfying $H \subset K \subset F$, $|K| = |H| + 1$. By the maximal choice of H, there is some $F(K) \in \mathcal{F}_t$ satisfying $F \cap F(K) = K$.

Let x be the unique element of $K - H$ and consider the $(k-1)$-element set $G = F(K) - \{x\}$. Then there exists some $F(K) \neq F' \in \mathcal{F}_t$ with $G \subset F'$. However, this implies $F \cap F' = H$ (this is the place where we use the fact that $\mathcal{F}$ is k-partite), a contradiction. ∎

5. Related and open problems.

Probably the most important open problem in this are is Conjecture 1. Unfortunately, Proposition 1 does not hold for k-uniform families in general. Otherwise one could give a proof using the following result.

Theorem ([4], [6]). *Suppose that $\mathcal{F} \subset \binom{X}{k}$ is ℓ-degenerate, $0 \leq \ell < k$. Then $|\mathcal{F}| \leq \binom{n-k+\ell}{\ell}$.*

What happens if one considers k-uniform families are not ℓ-dense?

An easy construction is the following. Fix a $(k-\ell+1)$-element set $B \subset X$, and define $\mathcal{F} = \{F \in \binom{X}{k} : B \subset F\}$. Clearly, $|\mathcal{F}| = \binom{n-k+\ell-1}{\ell-1}$ and $\mathcal{F}$ is not ℓ-dense.

This suggests:

Conjecture 2. *Suppose that $\mathcal{F} \subset \binom{X}{k}$ is not ℓ-dense, $k > \ell > 2$. Then*

$$|\mathcal{F}| \leq \binom{n-k+\ell-1}{\ell-1} \text{ holds for } n > n_0(k). \tag{7}$$

Again, the weaker bound $|\mathcal{F}| \leq \binom{n}{\ell-1}$ follows from the linear algebra argument of [5].

Recall that a family $\mathcal{F}$ is called *t-intersecting* if $|F \cap F'| \geq t$ holds for all F, $F' \in \mathcal{F}$.

Proposition 2. *If $\mathcal{F} \subset \binom{X}{k}$ is t-intersecting then it is not $(k-t+1)$-dense.*

Proof. Suppose for contradiction that $\mathcal{F}_B = 2^B$ sor some $(k-t+1)$-set B. Choose F, $F' \in \mathcal{F}$ to satisfy $F \cap B = B$ and $F' \cap B = \Phi$, respectively. Then $B \subset (F - F')$, implying $|F \cap F'| \leq |F| - |B| = t-1$, a contradiction. ∎

In view of Proposition 2, if true then Conjecture 2 would sharpen the Erdős-Ko-Rado Theorem, which says that (7) holds for $(k-\ell+1)$-intersecting families.

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